ON THE MAP OF BÖKSTEDT-MADSEN FROM THE COBORDISM CATEGORY TO A-THEORY

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Abstract. Bökstedt and Madsen defined an infinite loop map from the embedded d-dimensional cobordism category of Galatius, Madsen, Tillmann and Weiss to the algebraic K-theory of $BO(d)$ in the sense of Waldhausen. The purpose of this paper is to establish two results in relation to this map. The first result is that it extends the universal parametrized A-theory Euler characteristic of smooth bundles with compact d-dimensional fibers, as defined by Dwyer, Weiss and Williams. The second result is that it actually factors through the canonical unit map $Q(BO(d)_+) \to A(BO(d))$.

1. Introduction

The parametrized Euler characteristic was defined by Dwyer, Weiss and Williams in [6] for fibrations whose fibers are homotopy equivalent to a finite CW complex. Broadly speaking, the Euler characteristic of such a fibration $p : E \to B$ is a map that associates to every $b \in B$ the Euler class of the fiber $p^{-1}(b)$. The precise definition, which is given in terms of Waldhausen's algebraic K-theory of spaces (A-theory) [15], produces this way a section of an associated fibration

$$A_B(p) : A_B(E) \to B$$

that is defined by applying the A-theory functor to p fiberwise.

In the case where the fibration is actually a smooth fiber bundle and the fibers are compact smooth d-manifolds, possibly with boundary, the "Riemann-Roch theorem" of [6] asserts that this fiberwise Euler characteristic can be identified with the composition of a stable transfer map, in the sense of Becker and Gottlieb [2], followed by the "unit map" from stable homotopy to algebraic K-theory. More concretely, if we consider the vertical tangent bundle of the smooth fiber bundle $p : E \to B$ to pass to $BO(d)$, the parametrized A-theory Euler characteristic gives a map

$$\chi^{DWW} : B \to A(BO(d)).$$

According to the Riemann-Roch theorem, the diagram

$$\begin{tikzcd} B \arrow[r, "tr"] \arrow[dr, "\chi"'] & Q(BO(d)_+) \arrow[d, "\eta"] \\ & A(BO(d)) \end{tikzcd}$$

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is commutative up to homotopy, where the map tr is given by the classical Becker-Gottlieb transfer and η denotes the unit map at $BO(d)$.

Let $\mathcal{C}_d$ be the embedded d-dimensional cobordism category of [7]. Roughly speaking, the objects are closed smooth $(d-1)$-manifolds and the morphisms are cobordisms between them, all embedded in some high dimensional Euclidean space. Every closed smooth d-manifold M, embedded in some high dimensional Euclidean space, may be regarded as a cobordism from the empty manifold to itself and therefore it defines a loop in $B\mathcal{C}_d$. This rule defines a map

$$i_M : B\,\mathrm{Diff}(M) \to \Omega B\mathcal{C}_d$$

where $B\,\mathrm{Diff}(M)$ is the classifying space of smooth fiber bundles with fiber M. Recently, Bökstedt and Madsen [3] defined an infinite loop map

$$\tau \colon \Omega B\mathcal{C}_d \to A(BO(d))$$

which, in non-technical language, is given by viewing an n-simplex in the nerve of $\mathcal{C}_d$ as a filtered space equipped with a map to $BO(d)$ defined by the tangent bundle. This raises naturally the following two questions:

(1) Does the restriction of the map τ to $B\,\mathrm{Diff}(M)$ agree with the parametrized A-theory Euler characteristic of the universal bundle over $B\,\mathrm{Diff}(M)$?
(2) Does the map τ also factor through stable homotopy, via the unit map η, as in the Riemann-Roch theorem above?

Bökstedt and Madsen expressed their belief that the answer to both questions is affirmative in [3].

The purpose of this note is to show that both statements are indeed true. Assuming that (1) is true, then (2) can be regarded as a question about the additivity property of the parametrized A-theory Euler characteristic with respect to the fiber. Although a proof of (2) (assuming (1)) along these lines should be possible, ours follows a different route and is probably technically a lot simpler. The first main ingredient is to consider the cobordism category $\mathcal{C}_{d,\partial}$ of compact smooth manifolds with boundary, studied by Genauer [8], which contains $\mathcal{C}_d$ as a subcategory. The Bökstedt-Madsen map can be extended to a map

$$\tilde{\tau} \colon \Omega B\mathcal{C}_{d,\partial} \to A(BO(d)).$$

The space $\Omega B\mathcal{C}_{d,\partial}$ receives a map from $B\,\mathrm{Diff}(M)$, defined as before, for every M compact smooth d-manifold, possibly with boundary. In Theorem 5.4, we show that the restriction of $\tilde{\tau}$ to $B\,\mathrm{Diff}(M)$ agrees with the universal parametrized A-theory Euler characteristic, thus also answering Question (1). The proof uses the second main ingredient, namely, that the universal bundle over $B\,\mathrm{Diff}(M)$ defines a *bivariant A-theory characteristic* in the bivariant A-theory of the bundle (see [16]), and that the universal parametrized A-theory Euler characteristic is the image of this characteristic under a *coassembly map*. Since a basic problem in comparing all these maps is to find first the right identifications between the various models used to represent the various homotopy types, bivariant A-theory becomes extremely useful here, because it can offer a unifying perspective.

The homotopy type of $\Omega B\mathcal{C}_{d,\partial}$ was identified by Genauer [8] to be equivalent to $Q(BO(d)_+)$. To answer Question (2), we show in Theorem 5.6 that, under this identification, the map $\tilde{\tau}$ agrees with the unit map. This provides a geometric description of the unit map at $BO(d)$ in terms of smooth d-dimensional cobordisms. From this, it follows that the Bökstedt-Madsen map τ factors through the unit map.

Organization of the paper. In section 2, we recall the definitions of the cobordism categories $\mathcal{C}_d$ and $\mathcal{C}_{d,\partial}$ and state the main results about their homotopy types from [7] and [8] respectively. In section 3, we discuss the bivariant A-theory of a fibration and study some of its properties. Only very special instances of bivariant A-theory will appear in the proofs of the main results, however we hope that the discussion of section 3 will also be of independent interest. In section 4, we review the construction of the A-theory coassembly map and recall the definition of the parametrized A-theory Euler characteristic from [6], [16]. In section 5, we prove the main results of the paper, answering questions (1) and (2) above. In section 6, we end with a couple of remarks about the generalization of the Bökstedt-Madsen map to cobordism categories with tangential structures, and the connection with the work of Tillmann [12] where an analogous map was defined in the case of (a discrete version of) the oriented 2-dimensional cobordism category.

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2. The cobordism categories $\mathcal{C}_d$ and $\mathcal{C}_{d,\partial}$

In this section we recall the main results about the homotopy types of the embedded d-dimensional cobordism categories $\mathcal{C}_d$ and $\mathcal{C}_{d,\partial}$ from [7] and [8] respectively. For every $n \in \mathbb{N} \cup \{\infty\}$, there is a topological category $\mathcal{C}_{d,n}$ defined as follows. An object of $\mathcal{C}_{d,n}$ is a pair (M, a) where $a \in \mathbb{R}$ and M is a closed smooth $(d-1)$-dimensional submanifold of $\mathbb{R}^{d-1+n}$. (For $n = \infty$, define $\mathbb{R}^{d-1+\infty} := \operatorname{colim}_{n\to\infty} \mathbb{R}^{d-1+n}$ with the weak topology.) A non-identity morphism from (M_0, a_0) to (M_1, a_1) is a triple (W, a_0, a_1) where $a_0 < a_1$ and W is a compact smooth d-dimensional submanifold of $[a_0, a_1] \times \mathbb{R}^{d-1+n}$ such that for some $\epsilon > 0$, we have:

(i) $W \cap ([a_0, a_0 + \epsilon) \times \mathbb{R}^{d-1+n}) = [a_0, a_0 + \epsilon) \times M_0$
(ii) $W \cap ((a_1 - \epsilon, a_1] \times \mathbb{R}^{d-1+n}) = (a_1 - \epsilon, a_1] \times M_1$
(iii) $\partial W = W \cap (\{a_0, a_1\} \times \mathbb{R}^{d-1+n})$.

Composition is defined by taking the union of subsets of $\mathbb{R} \times \mathbb{R}^{d-1+n}$. The identities are formally added and regarded as "thin" product cobordisms. We abbreviate

$$\mathcal{C}_d := \mathcal{C}_{d,\infty} = \operatorname*{colim}_{n\to\infty} \mathcal{C}_{d,n}.$$

The topology is defined as follows. For technical reasons, we work here with the slightly modified model discussed in [7, Remarks 2.1(ii) and 4.5]. Let $\mathbb{R}^\delta$ denote the set of real numbers with the discrete topology and set

$$B_n(M) = \operatorname{Emb}(M, \mathbb{R}^{d-1+n}) / \operatorname{Diff}(M).$$

The space of objects $\mathrm{ob}\mathcal{C}_{d,n}$ is

$$\mathrm{ob}\mathcal{C}_{d,n} \cong \mathbb{R}^\delta \times \coprod_M B_n(M)$$

where M varies over the diffeomorphism classes of closed $(d-1)$-manifolds. By Whitney's embedding theorem, the space $\operatorname{Emb}(M, \mathbb{R}^{d-1+\infty})$ is contractible, and so there is a homotopy equivalence $B_\infty(M) \simeq B\operatorname{Diff}(M)$.

The definition of the topology on the morphisms is similar, but requires in addition that the collars are preserved under the diffeomorphisms. In detail, given

a cobordism (W, h_0, h_1) from M_0 to M_1 with collars $h_0 : [0,1) \times M_0 \to W$ and $h_1 : (0,1] \times M_1 \to W$, and $0 < \epsilon < 1/2$, let

$$\mathrm{Emb}_\epsilon(W, [0,1] \times \mathbb{R}^{d-1+n})$$

be the subspace of smooth embeddings that restrict to product embeddings on the ϵ-neighborhood of the collared boundary (see [7] for a more precise definition). This technical assumption is crucial in order to have a well-defined composition of morphisms. Set

$$\mathrm{Emb}(W, [0,1] \times \mathbb{R}^{d-1+n}) := \operatorname*{colim}_{\epsilon \to 0} \mathrm{Emb}_\epsilon(W, [0,1] \times \mathbb{R}^{d-1+n}).$$

Let $\mathrm{Diff}_\epsilon(W)$ denote the group of diffeomorphisms of W that restrict to product diffeomorphisms on the ϵ-neighborhood of the collared boundary. Set

$$\mathrm{Diff}(W) = \mathrm{Diff}(W, h_0, h_1) := \operatorname*{colim}_{\epsilon \to 0} \mathrm{Diff}_\epsilon(W).$$

There is a principal $\mathrm{Diff}(W)$-action on $\mathrm{Emb}(W, [0,1] \times \mathbb{R}^{d-1+n})$. Set

$$B_n(W) := \mathrm{Emb}(W, [0,1] \times \mathbb{R}^{d-1+n}) / \mathrm{Diff}(W).$$

Then the space of morphisms $\mathrm{mor}\,\mathcal{C}_{d,n}$ is

$$\mathrm{mor}\,\mathcal{C}_{d,n} \cong \mathrm{ob}\mathcal{C}_d \sqcup \coprod_W ((\mathbb{R}^2_+)^\delta \times B_n(W))$$

where $W = (W, h_0, h_1)$ varies over the diffeomorphism classes of d-dimensional cobordisms and $(\mathbb{R}^2_+)^\delta$ denotes the open half plane $\{(a_0, a_1) \colon a_0 < a_1\}$ with the discrete topology.

We will be mainly interested in the "stable" case $n = \infty$. We recall the main result of [7] that identifies the homotopy type of the classifying space $B\mathcal{C}_d$. Let $\mathrm{Gr}_d(\mathbb{R}^{d+k})$ be the Grassmannian of d-dimensional linear subspaces in $\mathbb{R}^{d+k}$ and consider the two standard bundles over it : the tautological d-dimensional vector bundle $\gamma_{d,k}$ and its k-dimensional complement $\gamma_{d,k}^\perp$. The spectrum $\mathrm{MTO}(d)$ is the Thom spectrum associated to the inverse of the tautological vector bundle $\gamma_d := \gamma_{d,\infty}$ over $\mathrm{Gr}_d(\mathbb{R}^{d+\infty})$, i.e.

$$\mathrm{MTO}(d)_{d+k} \colon = \mathrm{Th}(\gamma_{d,k}^\perp)$$

and the structure maps are induced, after taking Thom spaces, from the pullback diagrams, $k \geq d$,

$$\begin{tikzcd} \gamma_{d,k}^\perp \oplus \epsilon^1 \arrow[r] \arrow[d] & \gamma_{d,k+1}^\perp \arrow[d] \\ \mathrm{Gr}_d(\mathbb{R}^{d+k}) \arrow[r] & \mathrm{Gr}_d(\mathbb{R}^{d+k+1}) \end{tikzcd}$$

Theorem 2.1 (Galatius-Madsen-Tillmann-Weiss [7]). *There is a weak equivalence*

$$\alpha \colon B\mathcal{C}_d \xrightarrow{\sim} \Omega^{\infty-1}\mathrm{MTO}(d).$$

Following similar methods, Genauer generalized the results of [7] to cobordism categories of manifolds with corners [8]. We will be mainly interested in the special case of manifolds with boundary. For every $n \in \mathbb{N} \cup \{\infty\}$, there is a cobordism category $\mathcal{C}_{d,\partial,n}$ of smooth d-dimensional cobordisms between manifolds with boundary, nicely embedded in $\mathbb{R} \times \mathbb{R}^{d-1+n}$. The precise definition is analogous:

(i)′ an object is a pair (M, a) where $a \in \mathbb{R}^\delta$ and M is a smooth neat $(d-1)$-dimensional submanifold of $\mathbb{R}_+ \times \mathbb{R}^{d-2+n}$. (This model of "discrete cuts" is not considered in [8], however the same remarks as in [7, Remarks 2.1(ii) and 4.5] apply in this case as well.)

(ii)′ A non-identity morphism from (M_0, a_0) to (M_1, a_1) is a triple (W, a_0, a_1) where $a_0 < a_1$ and W is a smooth neat d-dimensional submanifold (with corners) of $[a_0, a_1] \times \mathbb{R}_+ \times \mathbb{R}^{d-2+n}$ satisfying (i)-(iii) as above; composition of morphisms is by taking the union of subsets.

(iii)′ The topology is defined similarly by the orbit spaces of the actions of diffeomorphisms on spaces of neat embeddings; see [8] for a precise definition.

We abbreviate $\mathcal{C}_{d,\partial} := \mathcal{C}_{d,\partial,\infty} = \underset{n\to\infty}{\mathrm{colim}} \mathcal{C}_{d,\partial,n}$.

Theorem 2.2 (Genauer [8]). *There is a weak equivalence*

$$\tilde{\alpha} : B\mathcal{C}_{d,\partial} \xrightarrow{\sim} \Omega^{\infty-1}\Sigma^\infty BO(d)_+.$$

Both weak equivalences are obtained as parametrized versions of the Pontryagin-Thom collapse map. We recall the description of this collapse map in the case of a single compact, possibly with boundary, smooth d-manifold M neatly embedded in $(\epsilon, 1-\epsilon) \times \mathbb{R}_+ \times \mathbb{R}^{d-2+N}$. This can be regarded as a morphism in $\mathcal{C}_{d,\partial}$, from the empty manifold to itself, and therefore it defines a loop in $B\mathcal{C}_{d,\partial}$. Then the map $\tilde{\alpha}$ gives a loop in $\Omega^{\infty-1}\Sigma^\infty BO(d)_+$ defined as follows. Consider the Pontryagin-Thom collapse map

$$(S^{d-1+N} \wedge (\mathbb{R}_+ \cup \{\infty\}), S^{d-1+N} \times \{0\}) \to (\mathrm{Th}(\nu_M), \mathrm{Th}(\nu_{\partial M}))$$

and the classifying map of the normal bundle

$$(\mathrm{Th}(\nu_M), \mathrm{Th}(\nu_{\partial M})) \to (\mathrm{MTO}(d)_{d+N}, \mathrm{MTO}(d-1)_{d-1+N}).$$

The cofiber of the inclusion of spectra $\Sigma^{-1}\mathrm{MTO}(d-1) \hookrightarrow \mathrm{MTO}(d)$ is equivalent to the spectrum $\Sigma^\infty(BO(d)_+)$ [7, Proposition 3.1]. So the composite map of pairs induces a stable map on cofibers,

$$\Sigma^\infty S^0 \to \Sigma^\infty(BO(d)_+)$$

which defines the image of $\tilde{\alpha}$ at the embedded manifold M. On the other hand, if $\partial M = \varnothing$, then the composite map is a loop in $\Omega^{\infty-1}\mathrm{MTO}(d)$,

$$S^{d+N} \to \mathrm{MTO}(d)_{d+N}$$

which defines the image of α at the embedded closed manifold M.

Note that there is an inclusion functor of cobordism categories $\mathcal{C}_d \hookrightarrow \mathcal{C}_{d,\partial}$. The induced map on (the loop spaces of) the classifying spaces can be identified with the map of spectra

$$\mathrm{MTO}(d) \to \Sigma^\infty(BO(d)_+)$$

defined by the canonical inclusion of Thom spaces $\mathrm{Th}(\gamma_{d,k}^\perp) \hookrightarrow \mathrm{Th}(\gamma_{d,k}^\perp \oplus \gamma_{d,k}) \cong S^{d+k} \wedge BO(d)_+$. We refer the reader to [8, Section 6] for more details.

3. Bivariant A-theory

Bivariant A-theory was defined by Bruce Williams [16]. A less general "untwisted" version can be discovered in unpublished work of Waldhausen. A variation of the latter was also considered by Bökstedt and Madsen [3].

The purpose of this section is to review and, for technical convenience, slightly modify Williams's definition of bivariant A-theory. This associates to a fibration $p\colon E \to B$ a *bivariant A-theory spectrum* $A(p)$ that has the following properties:

(a) If B is the one-point space, then $A(p) = A(E)$.

(b) For every fibration $q\colon V \to B$ and fiberwise map $f\colon E \to V$ over B, there is a natural push-forward map $f_*\colon A(p) \to A(q)$. Moreover, push-forward maps are homotopy invariant, i.e. if f is a homotopy equivalence, then so is f_*.

(c) For every pull-back square

$$\begin{tikzcd} E \times_B B' \arrow[r] \arrow[d, "p'"] & E \arrow[d, "p"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

there is a natural pull-back map $g^*\colon A(p) \to A(p')$. Moreover, pull-back maps are homotopy invariant, i.e. if $g : B' \to B$ is a homotopy equivalence, then so is g^*.

(d) Push-forward maps commute with pull-back maps, i.e. given maps q, f and g as above, the following diagram commutes

$$\begin{tikzcd} A(p) \arrow[r, "f_*"] \arrow[d, "g^*"] & A(q) \arrow[d, "g^*"] \\ A(p') \arrow[r, "f'_*"] & A(q') \end{tikzcd}$$

where q' is the pull-back of q along g and $f'\colon E \times_B B' \to V \times_B B'$ is the map induced by f.

The space $A(p)$ is the K-theory of a Waldhausen category of retractive spaces over E that are suitably related to the fibration p. As usual, we assume that all spaces are compactly generated and Hausdorff. The category $\mathcal{R}(E)$ of retractive spaces over E consists of all diagrams of spaces

$$E \overset{i}{\rightarrowtail} X \overset{r}{\to} E$$

where $r \circ i = \mathrm{id}_E$ and i is a cofibration. A morphism of retractive spaces is a map over and under E. The category $\mathcal{R}(E)$ becomes a Waldhausen category if we define cofibrations (resp. weak equivalences) to be those morphisms whose underlying map of spaces is a cofibration (resp. homotopy equivalence). Let $\mathcal{R}^{hf}(E) \subset \mathcal{R}(E)$ be the full subcategory of all objects (X, i, r) which are homotopy finite, i.e. which are weakly equivalent, in $\mathcal{R}(E)$, to an object (X', i', r') such that $(X', i'(E))$ is a finite relative CW-complex. This is a Waldhausen subcategory of $\mathcal{R}(E)$, whose K-theory, denoted by $A(E)$, is the algebraic K-theory of the space E [15].

For the definition of the bivariant A-theory of p, we consider those retractive spaces over E that define families of homotopy finite retractive spaces over the fibers of p, parametrized by the points of B.

Definition 3.1. Let $p\colon E \to B$ be a fibration. The category $\mathcal{R}^{hf}(p) \subset \mathcal{R}(E)$ is the full subcategory of all objects $E \overset{i}{\rightarrowtail} X \overset{r}{\to} E$ such that:

(i) the composite pr is a fibration, and

(ii) for each $b \in B$, the space $(pr)^{-1}(b)$ is homotopy finite as an object of $\mathcal{R}(p^{-1}(b))$ (with the obvious structure maps).

Proposition 3.2. *For every fibration $p : E \to B$, the category $\mathcal{R}^{hf}(p)$ is a Waldhausen subcategory of $\mathcal{R}(E)$. Moreover, it satisfies the "2-out-of-3" axiom (i.e. it is saturated in the terminology of* [15]*) and admits functorial factorizations of morphisms into a cofibration followed by a weak equivalence.*

Proof. Define a cofibration, resp. weak equivalence, in $\mathcal{R}^{hf}(p)$ to be a morphism which defines a cofibration, resp. weak equivalence, in $\mathcal{R}(E)$. Since $\mathcal{R}^{hf}(p) \subset \mathcal{R}(E)$ is a full subcategory which contains the zero object, it will suffice to show that $\mathcal{R}^{hf}(p)$ is closed under push-outs along a cofibration in $\mathcal{R}(E)$. Let

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, tail] & X_1 \arrow[d, tail] \\
X_2 \arrow[r] & X
\end{tikzcd}$$

be a push-out diagram of retractive spaces over E, such that $p \circ r_i \colon X_i \to B$ are fibrations, for $i = 0, 1, 2$, whose fibers are homotopy finite relative to the fibers of p. Then the induced map $p \circ r \colon X \to B$ is a fibration (see [9]), and there is a push-out diagram

$$\begin{tikzcd}
(pr_0)^{-1}(b) \arrow[r] \arrow[d, tail] & (pr_1)^{-1}(b) \arrow[d, tail] \\
(pr_2)^{-1}(b) \arrow[r] & (pr)^{-1}(b)
\end{tikzcd}$$

which shows that $(pr)^{-1}(b)$ defines an object of $\mathcal{R}^{hf}(p^{-1}(b))$, since this category is closed under taking such push-outs. The class of homotopy equivalences clearly satisfies the "2-out-of-3" axiom, so $\mathcal{R}^{hf}(p)$ is saturated. It remains to show the existence of factorizations of morphisms. These will be obtained by the mapping cylinder construction as usual. Let $f : (X, i_X, r_X) \to (Y, i_Y, r_Y)$ be a morphism in $\mathcal{R}^{hf}(p)$. Consider $(X \times I, j_0 i_X, r_X \pi_X)$ as an object of $\mathcal{R}^{hf}(E)$, where $j_0(x) = (x, 0)$ and $\pi_X(x, t) = x$. A cylinder object $\mathrm{Cyl}_E(X)$ for (X, i_X, r_X) is defined by the pushout square in $\mathcal{R}^{hf}(p)$

$$\begin{tikzcd}
E \times I \arrow[r, "\mathrm{proj}"] \arrow[d, tail, "i_X \times \mathrm{id}"] & E \arrow[d, tail] \\
X \times I \arrow[r, "q"] & \mathrm{Cyl}_E(X)
\end{tikzcd}$$

Then the mapping cylinder M_f of the map $f : (X, i_X, r_X) \to (Y, i_Y, r_Y)$ is defined by the push-out in $\mathcal{R}^{hf}(p)$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, tail, "qj_0"] & Y \arrow[d, tail] \\
\mathrm{Cyl}_E(X) \arrow[r, "u"] & M_f
\end{tikzcd}$$

and is denoted by (M_f, i', r'). Note that the fiber of $p \circ r' : M_f \to B$ at $b \in B$ fits in the push-out diagram

$$\begin{tikzcd} (pr_X)^{-1}(b) \arrow[r] \arrow[d, tail, "qj_0"] & (pr_Y)^{-1}(b) \arrow[d, tail] \\ \mathrm{Cyl}_{p^{-1}(b)}((pr_X)^{-1}(b)) \arrow[r] & (pr')^{-1}(b) \end{tikzcd}$$

By the universal property of push-outs, there is a canonical map $(M_f, i', r') \to (Y, i_Y, r_Y)$ which is also a homotopy equivalence. Then the standard factorization of the map $f : (X, i_X, r_X) \to (Y, i_Y, r_Y)$ as

$$(X, i_X, r_X) \xrightarrow{uqj_1} (M_f, i', r') \longrightarrow (Y, i_Y, r_Y)$$

defines functorial factorizations in $\mathcal{R}^{hf}(p)$ with the required properties. □

Remark 3.3. If $p\colon X \times B \to B$ is the trivial fibration, then the Waldhausen category $\mathcal{R}^{hf}(p)$ is closely related to the bivariant category denoted by $W(X, B)$ in [3]. Later on, this notation will be used here to denote the (classifying space of the) weak equivalences of $\mathcal{R}^{hf}(p)$. From now on, when we discuss about the homotopy type of a small category, we will often omit the classifying space functor "B", or simply replace it by "$|\cdot|$", in order to simplify the notation.

Definition 3.4. Let $p\colon E \to B$ be a fibration. The bivariant A-theory of p is defined to be the space $A(p) := K(\mathcal{R}^{hf}(p)) = \Omega|wS_\bullet\mathcal{R}^{hf}(p)|$.

Notice that if B is a point, then the categories $\mathcal{R}^{hf}(p)$ and $\mathcal{R}^{hf}(E)$ are the same, so we have $A(p) = A(E)$ in this case.

We now proceed to define the push-forward and pull-back maps. Let $q\colon V \to B$ be another fibration and $f\colon E \to V$ a fiberwise map, i.e. $q \circ f = p$. The push-forward along f defines an exact functor of Waldhausen categories

$$f_*\colon \mathcal{R}(E) \to \mathcal{R}(V), X \mapsto X \cup_E V.$$

This actually restricts to an exact functor

$$f_*\colon \mathcal{R}^{hf}(p) \to \mathcal{R}^{hf}(q)$$

between the corresponding Waldhausen subcategories. Indeed we have already remarked that if X, E, and V are fibered over B, then so is also the adjunction space $X \cup_E V$. Moreover, the fiber of $X \cup_E V$ over a point $b \in B$ is the adjunction space $X_b \cup_{E_b} V_b$ and hence it is homotopy finite over V_b when X_b is homotopy finite over E_b. Hence we obtain a map in K-theory,

$$f_*\colon A(p) \to A(q).$$

To define the pull-back maps, consider a pull-back square

(1)
$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"] & E \arrow[d, "p"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

There is a functor

$$g^*\colon \mathcal{R}^{hf}(p) \to \mathcal{R}^{hf}(p')$$

defined by sending a retractive space X over E to the pull-back $X' := X \times_B B'$. This defines a retractive space over E' and a fibration over B'. Also for each $b' \in B'$ the fiber $X'_{b'} \cong X_{g(b)}$ is homotopy finite as a retractive space over $E'_{b'} \cong E_{g(b)}$. This shows that the functor is well-defined. Moreover, it preserves pushouts, cofibrations (see [9]) and homotopy equivalences, so it defines an exact functor of Waldhausen categories. Hence we obtain a map in K-theory,

$$g^*\colon A(p) \to A(p').$$

Remark 3.5 (Naturality). In order to obtain strict naturality of these maps (and also to ensure that the size of the Waldhausen categories is small) we have to make certain additional assumptions. Fix, once and for all, a set $\mathcal{U}$ of cardinality $2^{|\mathbb{R}|}$. In the definition of an object (X, i, r) in $\mathcal{R}^{hf}(p)$, where $p\colon E \to B$, we additionally require that X is a set-theoretical subset of $E \amalg B \times \mathcal{U}$, such that

(i) the composite
$$E \xrightarrow{i} X \hookrightarrow E \amalg B \times \mathcal{U}$$
is the inclusion of E into the disjoint union, and

(ii) the following diagram is commutative:

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr, "p\circ r"'] & & E \amalg B \times \mathcal{U} \arrow[dl, "p \amalg \mathrm{proj}"] \\ & B & \end{tikzcd}$$

For a map $f\colon E \to V$ over B, the adjunction space $X \cup_E V$ is naturally a subset of $V \amalg B \times \mathcal{U}$ satisfying conditions (i) and (ii). On the other hand, suppose that we are given a pull-back square (1), then the pull-back $X \times_B B'$ is naturally a subset of $E' \amalg B' \times \mathcal{U}$. Using these conventions, both push-forward and pull-back are *strictly functorial* and *commute with each other*.

Proposition 3.6. *Let $p : E \to B$ and $q : V \to B$ be fibrations and $f : E \to V$ a fiberwise map over B. If f is a homotopy equivalence, then so are the induced push-forward maps $wS_n f_* : wS_n\mathcal{R}^{hf}(p) \to wS_n\mathcal{R}^{hf}(q)$ for all $n \geq 0$. In particular, the push-forward map $f_* : A(p) \to A(q)$ is also a homotopy equivalence.*

Proof. We show this first in the case where $f\colon E \overset{\simeq}{\hookrightarrow} V$ is a trivial cofibration by applying Cisinski's generalized Approximation theorem [5] (cf. [15, Theorem 1.6.7]). So it suffices to check that the exact functor $f_* : \mathcal{R}^{hf}(p) \to \mathcal{R}^{hf}(q)$ has the approximation properties (AP1) and (AP2) of [5, p. 512]. Indeed the Approximation theorem of [5, Proposition 2.14] shows then that $wS_n f_*$ is a homotopy equivalence for all $n \geq 0$ (see [5, Proposition 2.3, Lemme 2.13]).

Since f is a homotopy equivalence, then clearly $g : X \to Y$ (over E) is a homotopy equivalence if and only if $f_*(g) : X \cup_E V \to Y \cup_E V$ is a homotopy equivalence, so (AP1) is satisfied. Let (X, i_X, r_X) be an object of $\mathcal{R}^{hf}(p)$, (Y, i_Y, r_Y) an object of $\mathcal{R}^{hf}(q)$ and

$$u : f_*(X, i_X, r_X) = (X \cup_E V, i'_X, r'_X) \to (Y, i_Y, r_Y)$$

a map in $\mathcal{R}^{hf}(q)$. We factorize the retraction map r_Y into a trivial cofibration and a fibration

$$Y \overset{j}{\rightarrowtail} Y' \xrightarrow{q} V.$$

Clearly $(Y', i_{Y'} = j \circ i_Y, q)$ is an object of $\mathcal{R}^{hf}(q)$ and its restriction $(Y'_{|E}, i_{Y'}, q)$ over E is an object of $\mathcal{R}^{hf}(p)$. There is an adjoint map

$$v : (X, i_X, r_X) \to (Y'_{|E}, i_{Y'}, q)$$

in $\mathcal{R}^{hf}(p)$. Then we have a diagram in $\mathcal{R}^{hf}(q)$ as follows

$$\begin{tikzcd}
(X \cup_E V, i'_X, r'_X) \arrow[r, "u"] \arrow[d, "f_*(v)"] & (Y, i_Y, r_Y) \arrow[d, "j", "\simeq"'] \\
(Y'_{|E} \cup_E V, i_{Y'}, q) \arrow[r, tail, "\simeq"] & (Y', i_{Y'}, q)
\end{tikzcd}$$

and therefore (AP2) is also satisfied. This concludes the proof in the case where f is a trivial cofibration. The general case of an arbitrary homotopy equivalence follows from this by factorizing $f : E \xrightarrow{\simeq} V$ in the standard way as

$$\begin{tikzcd}
E \arrow[r, tail, "\simeq"] \arrow[dr, two heads, "p"'] & (E \times I) \cup_E V \arrow[r, shift left] \arrow[d] & V \arrow[l, tail, shift left, "\simeq"] \arrow[dl, two heads, "q"] \\
& B &
\end{tikzcd}$$

to reduce this general case to the case of trivial cofibrations. □

Corollary 3.7. *Let $p : E \to B$ and $q : V \to B$ be fibrations and $f, g : E \to V$ two fiberwise maps over B. If $f \simeq_B g$ are fiberwise homotopic over B, then $wS_n f_* \simeq wS_n g_* : wS_n \mathcal{R}^{hf}(p) \to wS_n \mathcal{R}^{hf}(q)$ are homotopic for all $n \geq 0$. Moreover, $f_* \simeq g_* : A(p) \to A(q)$ are also homotopic.*

Proof. It suffices to prove the statement for the inclusions at the endpoints

$$j_0, j_1 \colon E \to E \times I$$

regarded as fiberwise maps from p to the fibration $p' \colon E \times I \to B$. Both are split by the projection $\pi : E \times I \to E$ over B. By Proposition 3.6, the push-forward maps $wS_n(j_0)_*$ and $wS_n(j_1)_*$ are homotopy equivalences with homotopy inverse given by $wS_n \pi_*$. It follows that they are homotopic. The last statement can be shown similarly. □

Proposition 3.8. *Let $p : E \to B$ be a fibration and $g \colon B' \to B$ a map as in diagram (1). If g is a homotopy equivalence, then so are the induced pull-back maps $wS_n g^* : wS_n \mathcal{R}^{hf}(p) \to wS_n \mathcal{R}^{hf}(p')$ for all $n \geq 0$. In particular, the pull-back map $g^* : A(p) \to A(p')$ is also a homotopy equivalence.*

Proof. It is enough to show that if $i_0, i_1 \colon B \to B \times I$ are the inclusions at the endpoints, then the induced maps

$$i_0^*, i_1^* \colon w\mathcal{R}^{hf}(p \times \mathrm{id}_I) \to w\mathcal{R}^{hf}(p)$$

are homotopic. By Corollary 3.7, it suffices to show that the maps

$$(j_0)_* \circ i_0^*, (j_1)_* \circ i_1^* \colon w\mathcal{R}^{hf}(p \times \mathrm{id}_I) \to w\mathcal{R}^{hf}(q)$$

are homotopic. Let

$$\pi \colon w\mathcal{R}^{hf}(p \times \mathrm{id}_I) \to w\mathcal{R}^{hf}(q)$$

be the forgetful functor which views a fibration over $B \times I$ as one over B. Then there are natural weak equivalences of functors

$$(j_0)_* \circ i_0^* \xrightarrow{\simeq} \pi \xleftarrow{\simeq} (j_1)_* \circ i_1^*$$

which give the desired homotopy after geometric realization. Applying the same argument in each degree of the $S_\bullet$-construction finishes the proof. $\square$

As a consequence of the homotopy invariance of bivariant A-theory, we can define the *thick model* for A-theory as follows. This model will be needed in the proofs of the main results. We abbreviate

$$S_q W(X, B) := |w S_q \mathcal{R}^{hf} \begin{pmatrix} X \times B \\ \downarrow \\ B \end{pmatrix} |.$$

The thick model for $|w S_q \mathcal{R}^{hf}(X)|$ is defined to be the geometric realization of the simplicial space

$$S_q W(X, \Delta^\bullet) := \left[[n] \mapsto S_q W \begin{pmatrix} X \times \Delta^n \\ \downarrow \\ \Delta^n \end{pmatrix} \right]$$

where $\Delta^n = |\Delta^n_\bullet|$ denotes the standard topological n-simplex and the simplicial operations are induced by the pull-back maps. By Proposition 3.8, the inclusion of the zero-skeleton defines a homotopy equivalence

$$S_q W(X, \Delta^0) \xrightarrow{\simeq} |S_q W(X, \Delta^\bullet)| \, .$$

Passing to the loop spaces of the geometric realizations, we obtain a homotopy equivalence

$$A(X) \xrightarrow{\simeq} A_\Delta(X) := \Omega |[q], [n] \mapsto S_q W(X, \Delta^n)|.$$

4. The parametrized A-theory Euler characteristic

The purpose of this section is to review a description of the parametrized A-theory Euler characteristic of Dwyer, Weiss and Williams [6] using bivariant A-theory. Let $p \colon E \to B$ be a fibration with homotopy finite fibers. The retractive space $E \times S^0$ over E is an object of $\mathcal{R}^{hf}(p)$, so it defines a point

$$\chi(p) \in A(p)$$

called the *bivariant A-theory characteristic* of p. Williams observed in [16] that the parametrized A-theory characteristic of [6] is actually the image of $\chi(p)$ under a *coassembly map*.

In order to describe this coassembly map, we recall first some facts about homotopy limits of categories. Let **cat** denote the (2-)category of small categories. For every small category $\mathcal{I}$, the category $\mathbf{cat}^{\mathcal{I}}$ of $\mathcal{I}$-shaped diagrams in **cat** is enriched over **cat** as follows: If $\mathcal{F}, \mathcal{G} \colon \mathcal{I} \to \mathbf{cat}$ are two functors, then the natural transformations from $\mathcal{F}$ to $\mathcal{G}$ are the objects of a small category $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$. The morphism set in this category between two natural transformations $\eta, \theta \colon \mathcal{F} \to \mathcal{G}$ is given by

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(\eta, \theta) = \{H \colon \mathcal{F} \times \underline{[1]} \to \mathcal{G}; H_0 = \eta, H_1 = \theta\}$$

where $\underline{[1]}$ denotes the constant $\mathcal{I}$-diagram at the category $[1]$.

Definition 4.1. Let $\mathcal{I}$ be a small category and $\mathcal{G}\colon \mathcal{I} \to \mathbf{cat}$ an $\mathcal{I}$-shaped diagram of small categories. The *homotopy limit* of $\mathcal{G}$ is the category

$$\operatorname{holim} \mathcal{G} := \underline{\operatorname{Hom}}(\mathcal{I}/?, \mathcal{G})$$

where $\mathcal{I}/?\colon \mathcal{I} \to \mathbf{cat}$ is defined on objects by sending $i \in \operatorname{ob}\mathcal{I}$ to the over category $\mathcal{I}/i$.

Remark 4.2. The nerve of the homotopy limit of an $\mathcal{I}$-shaped diagram of small categories agrees with the homotopy limit of the associated $\mathcal{I}$-shaped diagram of the nerves as defined in [4]. However, this definition should not be confused with the notion of homotopy limit as the derived functor of limit on the category of $\mathcal{I}$-shaped categories.

Remark 4.3. If the functor $\mathcal{G}$ actually takes values in Waldhausen categories (and exact functors), then, by the naturality of the construction, there is a simplicial category $[n] \mapsto \operatorname{holim} wS_n\mathcal{G}$.

The following lemma is a straightforward exercise in the definition of the homotopy limit.

Lemma 4.4. *A functor $F\colon \mathcal{C} \to \operatorname{holim} \mathcal{G}$ determines and is determined by the following data:*

(i) *for each $i \in \mathcal{I}$, a functor $F_i\colon \mathcal{C} \to \mathcal{G}(i)$, and*

(ii) *for each morphism $u\colon i \to j$ in $\mathcal{I}$, a natural transformation $u^!$ from $\mathcal{G}(u) \circ F_i$ to F_j, such that $\mathrm{id}_i^! = \mathrm{id}$, and the following cocycle condition is satisfied: for every $v\colon j \to k$ in $\mathcal{I}$, we have*

$$(v \circ u)^! = v^! \circ \mathcal{G}(v)(u^!)$$

as natural transformations between functors $\mathcal{C} \to \mathcal{G}(k)$.

We can now define the coassembly map associated to the fibration $p : E \to B$. Assume that B is the geometric realization of a simplicial set $B_\bullet$. Let $\operatorname{simp}(B)$ denote the category of simplices of B: an object is a simplicial map $\sigma\colon \Delta^n_\bullet \to B_\bullet$, and a morphism from σ to $\tau\colon \Delta^k_\bullet \to B_\bullet$ is a simplicial map $\Delta^n_\bullet \to \Delta^k_\bullet$ making the obvious diagram commutative. We will normally avoid the distinction between the simplex σ and its geometric realization. Consider the functor

$$w\mathcal{R}^{hf}(E|_?)\colon\ \operatorname{simp}(B) \to \mathbf{cat}, \sigma \mapsto w\mathcal{R}^{hf}(E|_\sigma),$$

which is defined on morphisms by the push-forward maps. For every $\sigma \in \operatorname{simp}(B)$, there is a restriction functor

$$F_\sigma\colon w\mathcal{R}^{hf}(p) \to w\mathcal{R}^{hf}(\sigma^* p) \hookrightarrow w\mathcal{R}^{hf}(E|_\sigma)$$

which sends a retractive space X over E, which fibers over B, to its restriction over the simplex σ viewed as a retractive space over the corresponding restriction of E. If $u\colon \sigma \to \tau$ is a morphism in $\operatorname{simp}(B)$, then there is a natural transformation induced by the inclusion of restrictions

$$u^!\colon u_* F_\sigma \to F_\tau.$$

An easy check shows that the cocycle condition is satisfied. The same construction works when $\mathcal{R}^{hf}$ is replaced by $S_n\mathcal{R}^{hf}$, the n-th simplicial degree in Waldhausen's $S_\bullet$-construction. Thus, by the Lemma 4.4, we obtain (simplicial) functors

$$\alpha\colon w\mathcal{R}^{hf}(p) \to \operatorname*{holim}_{\operatorname{simp}(B)} w\mathcal{R}^{hf}(E|_?), \quad \alpha\colon wS_\bullet\mathcal{R}^{hf}(p) \to \operatorname*{holim}_{\operatorname{simp}(B)} wS_\bullet\mathcal{R}^{hf}(E|_?).$$

Remark 4.5. Again there is a technical point to consider. As it stands, the category $\mathcal{R}^{hf}(\sigma^* p)$ is *not* a subcategory of $\mathcal{R}^{hf}(E|_\sigma)$ since an object in the former category is a subset of $E|_\sigma \amalg \Delta^n \times \mathcal{U}$ while an object in the latter category is a subset of $E|_\sigma \amalg \mathcal{U}$. To deal with this, fix once and for all

- a set-theoretic embedding of each standard simplex Δ^n into $\mathcal{U}$, and
- a bijection $\mathcal{U} \times \mathcal{U} \to \mathcal{U}$.

Using this, we have $\Delta^n \times \mathcal{U} \subset \mathcal{U} \times \mathcal{U} \cong \mathcal{U}$ and we obtain a well-defined embedding of categories $\mathcal{R}^{hf}(\sigma^* p) \hookrightarrow \mathcal{R}^{hf}(E|_\sigma)$.

We make the following

Observation 4.6. For every functor $\mathcal{G}\colon \mathcal{I} \to \mathbf{cat}$, the geometric realization defines a map $|\cdot|\colon |\operatorname{holim} \mathcal{G}| \to \operatorname{holim} |\mathcal{G}|$. This map is adjoint to the simplicial map

$$N_k \operatorname{holim} \mathcal{G} \xrightarrow{|\cdot|} \operatorname{map}(\Delta^k, \operatorname{map}_{\mathcal{I}}(|\mathcal{I}/?|, |\mathcal{G}|)) = \operatorname{map}(\Delta^k, \operatorname{holim} |\mathcal{G}|),$$

using the standard model for $\operatorname{holim} |\mathcal{G}|$. If $\mathcal{G}$ takes values in Waldhausen categories, then similarly there is a map $|\cdot|\colon |\operatorname{holim} wS_\bullet \mathcal{G}| \to \operatorname{holim} |wS_\bullet \mathcal{G}|$. Moreover, by taking loop spaces, we obtain the map

$$\rho\colon \Omega |\operatorname{holim} wS_\bullet \mathcal{G}| \to \operatorname{holim} \Omega |wS_\bullet \mathcal{G}| = \operatorname{holim} K \circ \mathcal{G}.$$

Definition 4.7. The *A-theory coassembly map* is defined to be the composite map

$$\operatorname{coass}(p)\colon A(p) \xrightarrow{\Omega|\alpha|} \Omega |\operatorname*{holim}_{\operatorname{simp}(B)} wS_\bullet \mathcal{R}^{hf}(E|_?)| \xrightarrow{\rho} \operatorname*{holim}_{\operatorname{simp}(B)} A(E|_?).$$

The target of the coassembly map is again natural with respect to the covariant and contravariant operations induced respectively by the push-forward and pull-back maps. If $f\colon E \to V$ is a map between fibrations over B, then there is a natural transformation $\mathcal{R}^{hf}(E|_?) \to \mathcal{R}^{hf}(V|_?)$ inducing

$$f_*\colon \operatorname*{holim}_{\operatorname{simp}(B)} A(E|_?) \to \operatorname*{holim}_{\operatorname{simp}(B)} A(V|_?).$$

On the other hand, consider a pull-back diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"] & E \arrow[d, "p"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

and suppose that $g\colon B' \to B$ is the geometric realization of a simplicial map $g_\bullet$. So there is a functor $\operatorname{simp}(g)\colon \operatorname{simp}(B') \to \operatorname{simp}(B)$ and for every object σ of $\operatorname{simp}(B')$, there is a canonical isomorphism $E'|_\sigma \cong E|_{g\circ\sigma}$, since both spaces are just the pull-back of E along $g \circ \sigma$. Hence we obtain a natural isomorphism of functors

$$\operatorname{simp}(g)^* A(E|_?) \cong A(E'|_?)$$

defined on $\operatorname{simp}(B')$. Then we can define the pull-back operation as

$$g^*\colon \operatorname*{holim}_{\operatorname{simp}(B)} A(E|_?) \to \operatorname*{holim}_{\operatorname{simp}(B')} \operatorname{simp}(g)^* A(E|_?) \xrightarrow{\cong} \operatorname*{holim}_{\operatorname{simp}(B')} A(E'|_?),$$

where the first map is induced by base-change along the functor $\operatorname{simp}(g)$. An easy check shows that $(g \circ h)^* = h^* \circ g^*$. The following proposition, which will be important later on, is now obvious.

Proposition 4.8. *The A-theory coassembly map is natural with respect to the covariant and the contravariant operations.*

Corollary 4.9. *The A-theory coassembly map of $p : E \to B$ is a homotopy equivalence if B is contractible.*

Proof. This is obvious if B is a point, since then the coassembly map is essentially the identity map. Suppose that B is contractible. Let F be the fiber of $p : E \to B$ over a 0-simplex of B. By naturality, we have a commutative diagram

$$\begin{tikzcd} A\begin{pmatrix} E \\ \downarrow \\ B \end{pmatrix} \arrow[r] \arrow[d, "\simeq"] & \operatorname{holim}_{\mathrm{simp}(B)} A(E|_?) \arrow[d] \\ A(F) \arrow[r, equal] & A(F) \end{tikzcd}$$

where the vertical maps are given by restriction at the 0-simplex and the horizontal ones by the coassembly map. By the homotopy invariance of Proposition 3.8, the left-hand vertical arrow is a homotopy equivalence. Since the functor $A(E|_?)$ sends all morphisms to homotopy equivalences, its homotopy limit is homotopy equivalent to the space of sections of a fibration over $|\operatorname{simp}(B)|$. Under this identification, the right-hand vertical map corresponds to the evaluation of the section at the chosen base-point. Since $|\operatorname{simp}(B)| \simeq *$, this evaluation map is also a homotopy equivalence and therefore the result follows. □

We mention the following alternative description of the coassembly map in the special case of a trivial fibration $\pi_B : X \times B \to B$. This will be useful in the next section. To ease the notation, let us abbreviate

$$W(X, B) := S_1 W(X, B) = |w\mathcal{R}^{hf}\begin{pmatrix} X \times B \\ \downarrow \\ B \end{pmatrix}|.$$

Assume that B is the geometric realization of a simplicial set $B_\bullet$. For every $x \in W(X, B)$, pulling back along the inclusion of the simplices defines a simplicial map $x^* : B_\bullet \to W(X, \Delta^\bullet)$. Define the *scanning map* to be the map

$$\operatorname{scan}(X, B) : W(X, B) \to \operatorname{map}(B, |W(X, \Delta^\bullet)|)$$

which sends x to the geometric realization of the simplicial map x^*. The same construction at the level of A-theory yields a map

$$\operatorname{scan}(X, B)\colon A\begin{pmatrix} X \times B \\ \downarrow \\ B \end{pmatrix} \to \operatorname{map}(B, A_\Delta(X))$$

and the following diagram is commutative, where the vertical maps are given by "group completion",

$$\begin{tikzcd}
W(X,B) \arrow[r, "\mathrm{scan}(X,B)"] \arrow[d] & \mathrm{map}(B, |W(X,\Delta^\bullet)|) \arrow[d] \\
A\begin{pmatrix} X\times B \\ \downarrow \\ B \end{pmatrix} \arrow[r, "\mathrm{scan}(X,B)"] & \mathrm{map}(B, A_\Delta(X))
\end{tikzcd}$$

Lemma 4.10. *There is a homotopy commutative diagram*

$$\begin{tikzcd}
A\begin{pmatrix} X\times B \\ \downarrow \\ B \end{pmatrix} \arrow[rr, "\mathrm{scan}(X,B)"] \arrow[dr, "\mathrm{coass}(\pi_B)"'] & & \mathrm{map}(B, A_\Delta(X)) \arrow[dl, "\simeq"', "h"] \\
& \operatorname*{holim}_{\mathrm{simp}(B)} A(X\times ?) &
\end{tikzcd}$$

Proof. It is enough to show that there is a commutative diagram as follows, where the labelled arrows and the composite right-hand arrow are homotopy equivalences:

$$\begin{tikzcd}
A\begin{pmatrix} X\times B \\ \downarrow \\ B \end{pmatrix} \arrow[r] \arrow[d] & \mathrm{map}(B, A_\Delta(X)) \arrow[d] \\
\mathrm{holim}_{\mathrm{simp}(B)} A(X\times ?) \arrow[r] \arrow[d, "\simeq"] & \mathrm{map}(B, |[n] \mapsto \mathrm{holim}_{\mathrm{simp}(\Delta^n_\bullet)} A(X\times ?)|) \arrow[d] \\
\mathrm{map}(B, A(X)) \arrow[r, "\simeq"] & \mathrm{map}(B, |\sin^{\mathrm{top}} A(X)|)
\end{tikzcd}$$

The horizontal maps in the first and second line are given by the scanning construction and the upper vertical maps are defined by the coassembly map. Since the coassembly map is natural with respect to the base, the upper square is commutative.

In the lower line, $\sin^{\mathrm{top}} = \mathrm{map}(\Delta^\bullet, -)$ stands for the topological singular construction. The map from the middle to the lower line is given by the projection from $X\times ?$ to X and the identification of the homotopy limit of a constant diagram with a mapping space. Under this identification the lower line of the diagram is induced by the adjunction between geometric realization and $\sin^{\mathrm{top}}$. This map is a homotopy equivalence, in fact, it is homotopic to the map induced by the inclusion of the zero-simplices $A(X) = \mathrm{map}(\Delta^0, A(X))$ into $\sin^{\mathrm{top}} A(X)$ since both maps are split by the co-unit

$$|\sin^{\mathrm{top}} A(X)| \to A(X)$$

of the adjunction. To see that the composite right-hand arrow is also a homotopy equivalence, note that it is defined by the map of simplicial spaces

$$A\begin{pmatrix} X\times\Delta^n \\ \downarrow \\ \Delta^n\end{pmatrix} \xrightarrow{\simeq} \operatorname*{holim}_{\mathrm{simp}(\Delta^n_\bullet)} A(X\times ?) \xrightarrow{\simeq} \mathrm{map}(\Delta^n, A(X))$$

which is a homotopy equivalence in each simplicial degree by Corollary 4.9, and therefore so is also its geometric realization, since both simplicial spaces are good in the sense of [11]. □

Definition 4.11. Let $p\colon E\to B = |B_\bullet|$ be a fibration with homotopy finite fibers.

(i) The *bivariant A-theory characteristic* $\chi(p)\in A(p)$ is the point determined by the retractive space $E\times S^0$ over E, considered as an object of $\mathcal{R}^{hf}(p)$.
(ii) The *parametrized A-theory Euler characteristic* is the image of the bivariant A-theory characteristic under the coassembly map

$$\mathrm{coass}(p)\colon A(p)\to \operatorname*{holim}_{\mathrm{simp}(B)} A(E|_?).$$

5. The Bökstedt-Madsen map to A-theory

Bökstedt and Madsen [3] defined an infinite loop map $\tau : \Omega B\mathcal{C}_d \to A(BO(d))$. Broadly speaking, the map sends an n-tuple of composable d-dimensional cobordisms to the union of the cobordisms, regarded as a filtered space, together with the map to $BO(d)$ that classifies the tangent bundle (cf. [12]). To make this precise, they described the map as a simplicial map on the singular set of $N_\bullet\mathcal{C}_d$ to the thick model for the A-theory of $BO(d)$.

Similarly we define a map

$$\tilde\tau : \Omega B\mathcal{C}_{d,\partial} \to A(BO(d))$$

that extends τ along the map induced by the inclusion functor $\mathcal{C}_d \hookrightarrow \mathcal{C}_{d,\partial}$. The map $\tilde\tau$ is defined by a bisimplicial map

$$\tilde\tau_{p,q} : \sin_p N_q\mathcal{C}_{d,\partial,n} \to S_q\mathcal{R}^{hf}(\mathrm{Gr}_d(\mathbb{R}^{d+n}),\Delta^p)$$

by letting $n\to\infty$ and taking the loop spaces of the geometric realizations. Here $\sin_\bullet(-)$ denotes the simplicial set of singular simplices. A (smooth) p-simplex of $N_q\mathcal{C}_{d,\partial,n}$

$$\sigma : \Delta^p \to \mathcal{C}_{d,\partial,n}((M_0,a_0),(M_1,a_1))\times\cdots\times\mathcal{C}_{d,\partial,n}((M_{q-1},a_{q-1}),(M_q,a_q))$$

determines a (smoothly embedded) smooth fiber bundle over Δ^p

$$\begin{tikzcd} E[a_0,a_q] \arrow[r, hook] \arrow[d, "\pi"] & \Delta^p\times[a_0,a_q]\times\mathbb{R}_+\times\mathbb{R}^{d-2+n} \arrow[dl] \\ \Delta^p & \end{tikzcd}$$

together with a filtering by a sequence of codimension zero smooth sub-bundles over Δ^p,

$$E[a_0,a_1]\subseteq\cdots\subseteq E[a_0,a_q].$$

The classifying map of the vertical tangent bundle of π restricts to maps $\tan^v(\pi): E[a_0, a_i] \to \mathrm{Gr}_d(\mathbb{R}^{d+n})$ for every $i = 1, \dots, q$. This produces a filtered sequence of retractive spaces over $\mathrm{Gr}_d(\mathbb{R}^{d+n}) \times \Delta^p$ given by

$$\mathrm{Gr}_d(\mathbb{R}^{d+n}) \times \Delta^p \rightarrowtail E[a_0, a_i] \cup_{E(a_0)} \mathrm{Gr}_d(\mathbb{R}^{d+n}) \times \Delta^p \xrightarrow{r} \mathrm{Gr}_d(\mathbb{R}^{d+n}) \times \Delta^p$$

where

$$E(a_0) = E[a_0, a_i] \cap (\Delta^p \times \{a_0\} \times \mathbb{R}^{d-1+n})$$

fibers over Δ^p, and the retraction map on $E[a_0, a_i]$ is defined as follows

$$r_{E[a_0,a_i]} = (\tan^v(\pi), \pi).$$

The collection of these retractive spaces defines an object of $S_q\mathcal{R}^{hf}(\mathrm{Gr}_d(\mathbb{R}^{d+n}), \Delta^p)$.

The following lemma is immediate from the definitions.

Lemma 5.1. *For every $1 \leq n \leq \infty$, the maps $\{\tilde{\tau}_{p,q}\}_{p,q}$ define a bisimplicial map*

$$\tau_{\bullet,*} : \sin_\bullet N_*\mathcal{C}_{d,\partial,n} \to S_*\mathcal{R}^{hf}(\mathrm{Gr}_d(\mathbb{R}^{d+n}), \Delta^\bullet).$$

Setting $n = \infty$ and taking the loop spaces of the geometric realizations of these bisimplicial sets, we obtain a (weak) map:

$$\tilde{\tau} : \Omega B\mathcal{C}_{d,\partial} \xleftarrow{\sim} \Omega|\sin_\bullet N_\bullet\mathcal{C}_{d,\partial}| \xrightarrow{\tilde{\tau}} A_\Delta(BO(d)) \xleftarrow{\sim} A(BO(d)).$$

Note that $\tilde{\tau}$ is a map of loop spaces by definition. The following proposition is now obvious.

Proposition 5.2. *The following diagram commutes up to homotopy,*

$$\begin{tikzcd} \Omega B\mathcal{C}_d \arrow[rr, hook] \arrow[dr, "\tau"'] & & \Omega B\mathcal{C}_{d,\partial} \arrow[dl, "\tilde{\tau}"] \\ & A(BO(d)) & \end{tikzcd}$$

In view of Theorem 2.2, it follows that the map τ factors through $Q(BO(d)_+) := \Omega^\infty\Sigma^\infty BO(d)_+$. Moreover, it will be shown in Theorem 5.6 that the map $\tilde{\tau}$ can be identified with the canonical unit map $\eta : Q(BO(d)_+) \to A(BO(d))$.

Remark 5.3. Similarly we can define maps from other cobordism categories with corners to $A(BO(d))$ that in turn extend the map $\tilde{\tau}$ above. We refer the reader to [8, Definition 4.1] for the precise definition of these cobordism categories, and to [8, Proposition 6.1] for the general result determining their homotopy types in the unoriented case.

Let M be a compact smooth d-dimensional manifold, possibly with boundary, neatly embedded in $(\epsilon, 1-\epsilon) \times \mathbb{R}_+ \times \mathbb{R}^\infty$. This can be viewed as an (endo)morphism of the empty manifold in $\mathcal{C}_{d,\partial}$ (situated inside $\{0\} \times \mathbb{R}^\infty$ and $\{1\} \times \mathbb{R}^\infty$). Thus we obtain an inclusion map $i_M : B_\infty(M) \hookrightarrow N_1\mathcal{C}_{d,\partial} \to \Omega_\varnothing B\mathcal{C}_{d,\partial}$. Let χ_M^{BM} denote the restriction of the map $\tilde{\tau}$ along i_M, i.e.

$$\chi_M^{BM} := \tilde{\tau} \circ i_M.$$

Our first goal is to compare the map χ_M^{BM} with the universal parametrized A-theory Euler characteristic χ_M^{DWW} [6], [16]. Explicitly, the map χ_M^{BM} is defined as follows. Write $B_M = |\sin_\bullet B_\infty(M)|$ and let $p_M: E_M \to B_M$ be the universal smooth bundle with fiber M. The vertical tangent bundle defines a map over B_M,

$$\mathrm{Tan}^v(p_M): E_M \to B_M \times BO(d)$$

which induces a functor

$$\mathrm{Tan}^v(p_M)_*\colon w\mathcal{R}^{hf}\begin{pmatrix} E_M \\ \downarrow \\ B_M \end{pmatrix} \to w\mathcal{R}^{hf}\begin{pmatrix} BO(d)\times B_M \\ \downarrow \\ B_M \end{pmatrix}.$$

The retrative space $E_M \times S^0$ determines a point in $|w\mathcal{R}^{hf}(p_M)|$. After "group completion", this point is just the bivariant A-theory characteristic of p_M. Then the scanning construction applied to the image of this specific point under $\mathrm{Tan}^v(p_M)_*$, followed by "group completion", define the map:

$$\chi_M^{BM}\colon B_\infty(M) \simeq B_M \to |W(BO(d),\Delta^\bullet)| \simeq |w\mathcal{R}^{hf}(BO(d))| \to A(BO(d)).$$

As scanning is compatible with "group completion", the map χ_M^{BM} of Bökstedt-Madsen agrees with the image of $\mathrm{Tan}^v(p_M)_*(\chi(p_M))$ under the scanning construction

$$A\begin{pmatrix} BO(d)\times B_M \\ \downarrow \\ B_M \end{pmatrix} \to \mathrm{map}(B_M, A_\Delta(BO(d))),$$

once we have identified $A_\Delta(BO(d))$ with $A(BO(d))$ and $B_\infty(M)$ with B_M.

On the other hand, we obtain a new map by passing to the parametrized A-theory Euler characteristic of p_M first, via the coassembly map, and then applying $\mathrm{Tan}^v(p_M)_*$ to it. In this case, the image of the parametrized A-theory Euler characteristic of p_M under the composite map

$$\operatorname*{holim}_{\mathrm{simp}(B_M)} A(E_M|_?) \xrightarrow{\mathrm{Tan}^v(p_M)_*} \operatorname*{holim}_{\mathrm{simp}(B_M)} A(BO(d)\times ?) \overset{h}{\simeq} \mathrm{map}(B_M, A_\Delta(BO(d)))$$

defines the universal parametrized A-theory Euler characteristic map for smooth bundles with fiber M (cf. [16], [6]),

$$\chi_M^{DWW}\colon B_\infty(M) \simeq B_M \to A_\Delta(BO(d)) \simeq A(BO(d)).$$

Theorem 5.4. *The maps χ_M^{BM} and χ_M^{DWW} agree up to homotopy.*

Proof. Let $\tilde\chi$ denote the image of $\chi(p_M)$ under the push-forward of $\mathrm{Tan}^v(p_M)$,

$$\mathrm{Tan}^v(p_M)_*\colon w\mathcal{R}^{hf}\begin{pmatrix} E_M \\ \downarrow \\ B_M \end{pmatrix} \to w\mathcal{R}^{hf}\begin{pmatrix} BO(d)\times B_M \\ \downarrow \\ B_M \end{pmatrix}.$$

By Proposition 4.8, the coassembly map commutes with the push-forward map $\mathrm{Tan}^v(p_M)_*$. Hence the image of the parametrized A-theory characteristic of p_M, under the push-forward map

$$\operatorname{holim}\mathrm{Tan}^v(p_M)_*\colon \operatorname*{holim}_{\mathrm{simp}(B_M)} A(E|_?) \to \operatorname*{holim}_{\mathrm{simp}(B_M)} A(BO(d)\times ?)$$

agrees with the image of $\tilde\chi$, under the coassembly map. By definition, this point defines the homotopy class of χ_M^{DWW} via the homotopy equivalence h. On the other hand, the homotopy class of χ_M^{BM} is the component of the image of $\tilde\chi$ under the scanning map. Thus we have the following diagram

$$\begin{tikzcd}
\chi(p_M) \arrow[r, mapsto, "\mathrm{Tan}^v(p_M)_*"] \arrow[d, mapsto, "\mathrm{coass}"] & \tilde\chi \arrow[r, mapsto, "\mathrm{scan}"] \arrow[d, mapsto, "\mathrm{coass}"] & {[\chi_M^{BM}]} \arrow[d, equal] \\
\mathrm{coass}(\chi(p_M)) \arrow[r, mapsto, "\operatorname{holim}\mathrm{Tan}^v(p_M)_*"] & \mathrm{coass}(\tilde\chi) \arrow[r, leftrightarrow, "h"] & {[\chi_M^{DWW}]}
\end{tikzcd}$$

and the agreement of the two homotopy classes of maps in $\pi_0 \operatorname{map}(B_M, A(BO(d)))$ follows from the homotopy commutative diagram of Lemma 4.10. $\square$

The weak equivalence of Theorem 2.2 implies that $\Omega B\mathcal{C}_{d,\partial}$ admits the structure of an infinite loop space. Broadly speaking, this is the same structure as the one induced by the operation of making two embedded cobordisms disjoint and taking their disjoint union. However, some careful analysis is required to make this operation precise since there is no canonical choice of making two embedded cobordisms disjoint, in a symmetric manner. A possible approach is to construct a Γ-space consisting of n-tuples of cobordisms that are disjoint. Another one would be to follow the methods of [3] to construct deloopings of $B\mathcal{C}_{d,\partial}$ geometrically. For our purposes here, it will suffice to assume the infinite loop space structure on $\Omega B\mathcal{C}_{d,\partial}$ that is induced by $Q(BO(d)_+)$.

Let D^d denote the d-dimensional closed disk. The inclusion of linear diffeomorphisms $O(d) \to \operatorname{Diff}(D^d)$ induces a map on classifying spaces $BO(d) \to B\operatorname{Diff}(D^d) \simeq B_\infty(D^d)$. More generally, if D^d_n denotes a disjoint union of n copies of D^d, then there is a map

$$\phi_n \colon X_n := BO(d)^n \times_{\Sigma_n} \operatorname{Emb}(\{1, \cdots, n\}, \mathbb{R}^\infty) \to B_\infty(D^d_n),$$

from the space of configurations of n points in $\mathbb{R}^\infty$ labelled by d-dimensional linear subspaces in $\mathbb{R}^\infty$ to the space of configurations of n disjoint d-disks in $\mathbb{R}^\infty$. The disjoint union of the spaces X_n, for all n, can be adjusted up to weak equivalence to form a topological monoid whose group completion is homotopy equivalent to $Q(BO(d)_+)$, see [10].

Lemma 5.5. *The composite*

$$\coprod_{n=1}^{\infty} X_n \xrightarrow{\coprod_{n=1}^{\infty} \phi_n} \coprod_{n=1}^{\infty} B_\infty(D^d_n) \xrightarrow{\coprod_{n=1}^{\infty} i_{D^d_n}} \Omega B\mathcal{C}_{d,\partial}$$

is a group completion. Moreover, the induced weak equivalence $\beta \colon Q(BO(d)_+) \to \Omega B\mathcal{C}_{d,\partial}$ is a homotopy inverse to $\tilde{\alpha}$ of Theorem 2.2.

Proof. Recall that the weak equivalence $\tilde{\alpha}$ of Theorem 2.2 (as well as the map α of Theorem 2.1) is given by a Pontryagin-Thom collapse map applied to every embedded cobordism (see also section 2). The map of the Lemma followed by the weak equivalence $\tilde{\alpha}$ can be identified with the group completion map

$$\coprod_{n=1}^{\infty} BO(d)^n \times_{\Sigma_n} \operatorname{Emb}(\{1, \cdots, n\}, \mathbb{R}^\infty) \to Q(BO(d)_+)$$

since this can also be described by a collapse map [10, §3]. $\square$

Let $\eta \colon Q(BO(d)_+) \to A(BO(d))$ be the unit transformation of A-theory evaluated at $BO(d)$. This factors through the excisive approximation to A-theory represented by the spectrum $A(*)$, e.g. see [6, Section I.4]. For a more geometric construction, following Waldhausen's manifold approach [14], see [1].

Theorem 5.6. *The following diagram commutes up to homotopy,*

$$\begin{tikzcd} \Omega B\mathcal{C}_{d,\partial} \arrow[rr, "\tilde{\alpha}", "\sim"'] \arrow[dr, "\tilde{\tau}"'] & & Q(BO(d)_+) \arrow[dl, "\eta"] \\ & A(BO(d)) & \end{tikzcd}$$

Proof. Since η and $\tilde{\tau}$ are both loop maps, it suffices to show that the diagram commutes up to homotopy when restricted along $i_{D^d_n} \circ \phi_n \colon X_n \to \Omega B\mathcal{C}_{d,\partial}$, for every n. Since $i_{D^d_n} \circ \phi_n$ factors through $B_\infty(D^d_n)$, it suffices to show that, for every $n \geq 1$, the following diagram commutes up to homotopy:

$$\begin{tikzcd} B_\infty(D^d_n) \arrow[rr, "\tilde{\alpha}\circ i_{D^d_n}"] \arrow[dr, "\chi^{BM}_{D^d_n}"'] & & Q(BO(d)_+) \arrow[dl, "\eta"] \\ & A(BO(d)) & \end{tikzcd}$$

By Theorem 5.4, $\chi^{BM}_{D^d_n}$ is homotopic to $\chi^{DWW}_{D^d_n}$. Also the composite map $\tilde{\alpha} \circ i_{D^d_n}$ is homotopic to the Becker-Gottlieb transfer map of the universal D^d_n-bundle, followed by the map of the vertical tangent bundle. Then the commutativity of the diagram follows from the universal Riemann-Roch theorem of [6, Theorem 8.5]. □

Remark 5.7. Theorems 5.4 and 5.6 can be seen as structured forms of an additivity property for the parametrized A-theory Euler characteristic and its factorization via the unit map. Note that while we have only assumed the universal Riemann-Roch theorem [6] for smooth bundles of disks, the combination of the two theorems implies the factorization

$$\begin{tikzcd} & Q(BO(d)_+) \arrow[d, "\eta"] \\ B\,\mathrm{Diff}(M) \arrow[ur, "tr"] \arrow[r, "\chi^{DWW}_M"'] & A(BO(d)) \end{tikzcd}$$

for every smooth d-manifold M (possibly with boundary).

6. Concluding Remarks

1. Similar ideas apply in the case of cobordism categories with tangential structures. Let $\theta : \mathbf{X} \to BO(d)$ be a fibration. The authors of [7] defined a cobordism category $\mathcal{C}^\theta_d$ of manifolds equipped with a tangential θ-structure, i.e. a lift of the map classifying the stable tangent bundle to $\mathbf{X}$. The main theorem of [7] identifies the homotopy type of $B\mathcal{C}^\theta_d$ with the infinite loop space $\Omega^{\infty-1}\mathrm{MT}\theta$ of the Thom spectrum associated with the stable bundle $\theta^*(-\gamma_d)$. Genauer [8] studied the cobordism categories $\mathcal{C}^\theta_{d,\partial}$ of manifolds with boundary and a tangential θ-structure and determined their homotopy types. We can similarly define a map

$$\tilde{\tau}^\theta : \Omega B\mathcal{C}^\theta_{d,\partial} \to A(X).$$

An important special case is the oriented cobordism category $\mathcal{C}^+_{d,\partial}$ associated to θ being the orientation cover. Then similar arguments show that there is a homotopy

commutative diagram as follows

$$\begin{tikzcd}
\Omega B\mathcal{C}^+_{d,\partial} \arrow[r, "\sim"] & \Omega^\infty\Sigma^\infty BSO(d)_+ \arrow[r, "\eta"] & A(BSO(d)) \\
\Omega B\mathcal{C}^+_d \arrow[u] \arrow[urr, "\tau^+"'] & &
\end{tikzcd}$$

The weak equivalence in the diagram is shown in [8, Proposition 6.2].

2. A version of the Bökstedt-Madsen map in the oriented 2-dimensional case was defined in [12]. This map was used there to deduce the existence of a certain splitting of the homotopy type of that cobordism category. The arguments apply similarly in higher dimensions. Let M be a closed d-dimensional manifold embedded in $\mathbb{R}^\infty$, so that it may be regarded as a (endo)morphism in $\mathcal{C}_d$. Thus it defines a point in $\Omega B\mathcal{C}_d$ and, using the infinite loop space structure, we can extend the inclusion of this point to an infinite loop map

$$j_M \colon QS^0 \to \Omega B\mathcal{C}_d.$$

By composing j_M with the composite infinite loop map

$$\Omega B\mathcal{C}_d \xrightarrow{\tau} A(BO(d)) \xrightarrow{e_*} A(*) \xrightarrow{Tr} QS^0$$

where $e : BO(d) \to *$ and Tr denotes Waldhausen's trace map [13], we obtain a self map of QS^0. By Theorem 5.4, it is easy to see that the homotopy class of this map can be identified with the Euler characteristic of M, $\chi(M) \in \pi_0^s \cong \mathbb{Z}$. Thus, for every such M, we obtain a geometric description of a splitting of a copy of the localized sphere spectrum $(QS^0)[\chi(M)^{-1}]$ from $\Omega B\mathcal{C}_d$, as infinite loop spaces.

These splittings can also be realized at the level of the Thom spectrum $\mathrm{MTO}(d)$ as follows. The bordism class of M defines an element $[M] \in \pi_0\mathrm{MTO}(d)$ represented by a map $QS^0 \to \Omega^\infty\mathrm{MTO}(d)$. Up to the weak equivalence α of Theorem 2.1, this is the same map as j_M. Composition with the map $\Omega^\infty\mathrm{MTO}(d) \to Q(BO(d)_+)$, given by the addition of the tautological bundle, and the map $Q(BO(d)_+) \to QS^0$, which collapses $BO(d)$ to a point, produces the same self-map of QS^0, specified as multiplication by $\chi(M)$. If $M \subseteq \mathbb{R}^N$, this is represented by the composite

$$S^N \to \mathrm{Th}(\nu_M) \to \mathrm{Th}(\gamma^\perp_{d,N-d}) \to \mathrm{Th}(\gamma^\perp_{d,N-d} \oplus \gamma_{d,N-d}) \cong S^N \wedge \mathrm{Gr}_d(\mathbb{R}^N)_+ \to S^N$$

where the first map is the Pontryagin-Thom collapse map, the second map is defined by the classifying map for the normal bundle of M, the third map is the addition of the tautological bundle and the fourth map is given by collapsing at the basepoint.

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